MATH 3063 ABSTRACT ALGEBRA Project 2 **Solutions**
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Definition 1. Let $f : A \to B$.
We say that f is *injective* if

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2.$$

We say that f is *surjective* if

$$\forall b \in B\, \exists a \in A \text{ such that } f(a) = b.$$

We say that f is *bijective* if f is injective and surjective.

Problem 1. Let $f : A \to B$ and $g : B \to C$. Suppose that f is surjective and $g \circ f$ is injective. Show that g is injective.

Proof. Let $b_1, b_2 \in B$ such that $g(b_1) = g(b_2)$. We wish to show that $b_1 = b_2$. Since f is surjective, there exist $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_1$. Applying g to these equations gives $g(f(a_1)) = g(b_1)$ and $g(f(a_2)) = g(b_2)$. But $g(b_1) = g(b_2)$, and since $g \circ f$ is injective, $a_1 = a_2$. Thus $f(a_1) = f(a_2)$, that is, $b_1 = b_2$. Therefore g is injective. □

Problem 2. Let $f : A \to B$ and $g : B \to C$. Suppose that g is injective and $g \circ f$ is surjective. Show that f is surjective.

Proof. Let $b \in B$. We wish to find $a \in A$ such that $f(a) = b$. Let $c = g(b)$. Since $g \circ f$ is surjective, there exists $a \in A$ such that $g(f(a)) = c$, that is, $g(f(a)) = g(b)$. Since g is injective, $f(a) = b$. Therefore f is surjective. □

Problem 3. Consider the relationship between composition and bijectivity.

(a) Show that the composition of injective functions is injective.

(b) Show that the composition of surjective functions is surjective.

(c) Conclude that the composition of bijective functions is bijective.

Solution.
(a) Let $f : A \to B$ and $g : B \to C$ be injective functions; we wish to show that $g \circ f : A \to C$ is injective.
Let $a_1, a_2 \in A$ such that $g \circ f(a_1) = g \circ f(a_2)$, that is, $g(f(a_1)) = g(f(a_2))$. Since g is injective, $f(a_1) = f(a_2)$. Since f is injective, $a_1 = a_2$. Thus $g \circ f$ is injective.
(b) Let $f : A \to B$ and $g : B \to C$ be surjective functions; we wish to show that $g \circ f : A \to C$ is surjective.
Let $c \in C$. Since g is surjective, there exists $b \in B$ such that $g(b) = c$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Thus $g \circ f(a) = g(f(a)) = g(b) = c$. Thus $g \circ f$ is surjective.
(c) Let $f : A \to B$ and $g : B \to C$ be bijective functions; we wish to show that $g \circ f : A \to C$ is bijective.
Since f and g are bijective, they are both injective and surjective. By **(a)**, $g \circ f$ is injective, and by **(b)**, $g \circ f$ is surjective. Thus $g \circ f$ is bijective. □

Definition 2. Let $f : A \to B$ and $g : B \to A$.

We say that g is an *inverse* of f if $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.
We say that g is a *left inverse* of f if $g \circ f = \mathrm{id}_A$.
We say that g is a *right inverse* of f if $f \circ g = \mathrm{id}_B$.
We say that f is *invertible* if the exists an inverse for f.
We say that f is *left invertible* if the exists a left inverse for f.
We say that f is *right invertible* if the exists a right inverse for f.

Problem 4. Consider the existence of left and right inverses by giving examples.

(a) Give an example of a function which is left invertible but not invertible.

(b) Give an example of a function which is right invertible but not invertible.

Solution.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \arctan x$. Then f is injective but not surjective, and so by Problem 5a is left invertible but not invertible.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 - x$. Then f is surjective but not injective, and so by Problem 5b, is right invertible but not invertible. □

Problem 5. Consider the relationship between invertibility and bijectivity.

(a) Show that a function is left invertible if and only if it is injective.

(b) Show that a function is right invertible if and only if it is surjective.

(c) Conclude that a function is invertible if and only if it is bijective.

Solution. Let $f : A \to B$.

(a) We show by directions of the double implication.

($\Rightarrow$) Suppose that f is left invertible, and let $g : B \to A$ be a left inverse, so that $g \circ f = \mathrm{id}_A$. Let $a_1, a_1 \in A$ such that $f(a_1) = f(a_2)$. Applying g to both sides gives $g(f(a_1)) = g(f(a_2))$, so $\mathrm{id}_A(a_1) = \mathrm{id}_A(a_2)$; that is, $a_1 = a_2$. Thus f is injective.

($\Leftarrow$) Suppose that f is injective. Then for every $b \in f(A)$ there exists a unique $a_b \in A$ such that $f(a_b) = b$. Let $a_0 \in A$. Define $g : B \to A$ by

$$g(b) = \begin{cases} a_b & \text{if } b \in f(A); \\ a_0 & \text{otherwise.} \end{cases}$$

Then $g(f(a)) = a$ for all $a \in A$. Thus g is a left inverse for f.

(b) We show by directions of the double implication.

($\Rightarrow$) Suppose that f is right invertible, and let $g : B \to A$ be a right inverse, so that $f \circ g = \mathrm{id}_B$. Let $b \in B$. Let $a = g(b)$. Then $f(g(b)) = b$, that is, $f(a) = b$.

($\Leftarrow$) Suppose that f is surjective. Then for every $b \in B$, we select an element $a_b \in A$ such that $f(a_b) = b$. Define $g : B \to A$ by $g(b) = a_b$. Then $f(g(b)) = f(a_b) = b$ for all $b \in B$, and g is a right inverse for f.

(c) Now

$$\begin{aligned} f \text{ is invertible} &\Leftrightarrow f \text{ is left invertible and } f \text{ is right invertible} \\ &\Leftrightarrow f \text{ is injective and } f \text{ is surjective} \\ &\Leftrightarrow f \text{ is bijective.} \end{aligned}$$

□

Definition 3. Let $P, Q \in \mathbb{R}^2$ be given by $P = (x_1, y_1)$ and $Q = (x_1, y_2)$. The *distance* from P to Q is

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

An *isometry* of $\mathbb{R}^2$ is a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $d(f(P), f(Q)) = d(P, Q)$.
Three types of isometries are translations, rotations, and reflections.
A translation is described by (h, k), where $(x, y) \mapsto (x + h, y + k)$.
A rotation is described by (a, b, θ), where (a, b) is a fixed point and θ is the angle of rotation.

Problem 6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry such that $f = T \circ R$, where T is the translation described by (h, k) and R is the rotation described by (a, b, θ). Suppose $f(5, 0) = (1, 2)$ and $f(7, 0) = (2, 2 + \sqrt{3})$. Find h, k, a, b, θ.

Solution. Note that the distance from $(1, 2)$ to $(2, 2 + \sqrt{3})$ is 2, which is the same as the distance between $(5, 0)$ and $(7, 0)$. So, some isometry takes the first pair to the second.

Let L be the line through $(5, 0)$ and $(7, 0)$, and let M be the line through $(1, 2)$ and $(2, 2 + \sqrt{3})$. Then $L : y = 0$ and $M : y = \sqrt{3}x + 2 - \sqrt{3}$. The angle between these lines in $60°$, and the intersection of these lines is at the point $A = (1 - \frac{2}{\sqrt{3}}, 0)$.

Let R be rotation about the point A by $60°$; then $R(L) = M$. Let T be translation by the vector $R(5, 0) - (1, 2)$. Then $TR(5, 0) = (1, 2)$ and $TR(7, 0) = (2, 2 + \sqrt{3})$. □